

Analytical solutions to some optimization problems on ranks and inertias of matrix-valued functions subject to linear matrix inequalities

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Abstract. Matrix rank and inertia optimization problems are a class of discontinuous optimization problems, in which the decision variables are matrices running over certain feasible matrix sets, while the ranks and inertias of the variable matrices are taken as integer-valued objective functions. In this paper, we establish a group of explicit formulas for calculating the maximal and minimal values of the rank- and inertia-objective functions of the Hermitian matrix expression $A_1 - B_1 X B_1^*$ subject to the linear matrix inequality $B_2 X B_2^* \succcurlyeq A_2$ ($B_2 X B_2^* \preccurlyeq A_2$) in the Löwner partial ordering, and give applications of these formulas in characterizing behaviors of some constrained matrix-valued functions.

Key Words: Matrix-valued function; matrix equation; LMI; rank; inertia; integer-valued objective function; feasible matrix set; generalized inverses of matrices; optimization; Löwner partial ordering

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1 Introduction

Throughout this paper,

$\mathbb{C}^{m \times n}$ stands for the set of all $m \times n$ complex matrices;

\mathbb{C}_H^m stands for the set of all $m \times m$ complex Hermitian matrices;

A^* , $r(A)$ and $\mathcal{R}(A)$ stand for the conjugate transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m \times n}$, respectively;

I_m denotes the identity matrix of order m ;

$[A, B]$ denotes a row block matrix consisting of A and B ;

the Moore–Penrose inverse of $A \in \mathbb{C}^{m \times n}$, denoted by A^\dagger , is defined to be the unique solution X satisfying the four matrix equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$;

the symbols E_A and F_A stand for $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$;

$i_+(A)$ and $i_-(A)$, called the partial inertia of $A \in \mathbb{C}_H^m$, are defined to be the numbers of the positive and negative eigenvalues of A counted with multiplicities, respectively;

$A \succcurlyeq 0$ ($A \succ 0$) means that A is Hermitian positive semi-definite (Hermitian positive definite);

two $A, B \in \mathbb{C}_H^m$ are said to satisfy the inequality $A \succcurlyeq B$ ($A \succ B$) in the Löwner partial ordering if $A - B$ is positive semi-definite (positive definite).

The matrix approximation problem is to approximate optimally, with respect to some criteria, a matrix by one of the same dimension from a given feasible matrix set. Assume that A is a matrix to be approximated. Then a conventional statement of general matrix optimization problems of A from this point of view can be written as

$$\text{minimize } \rho(A - Z) \quad \text{subject to } Z \in \mathcal{S}, \quad (1.1)$$

where $\rho(\cdot)$ is certain objective function, which is usually taken as the determinant, trace, norms, rank, inertia of matrix, and \mathcal{S} is a given feasible matrix set. A best-known case of (1.1) is to minimize the norm $\|A - Z\|_F^2$ subject to $Z \in \mathcal{S}$.

In this paper, we assume that the objective function $\rho(\cdot)$ in (1.1) is taken as the rank or inertia of matrix, and $A \in \mathbb{C}_H^{m \times m}$ is a Hermitian matrix. The rank and inertia of matrix, as objective functions, are often used when finding feasible matrices Z such that resulting $A - Z$ attains its maximal possible rank or inertia (is nonsingular or definite when square), or finding feasible matrix Z such that $A - Z$ attains the minimal rank or inertia as possible (called low-rank or low-inertia matrix completion). This kind of problems are usually called the matrix rank-optimization and inertia-optimization problems, or matrix rank and inertia completion problems in the literature. Generally speaking, matrix rank and inertia optimization problems are a class of discontinuous optimization problems, in which the decision variables are matrices running over certain matrix

sets, while the ranks and inertias of the variable matrices are taken as integer-valued objective functions. This kind of optimization problems can generally be written as

$$\text{maximize } r(A - Z) \quad \text{subject to } Z \in \mathcal{S}, \quad (1.2)$$

$$\text{minimize } r(A - Z) \quad \text{subject to } Z \in \mathcal{S}, \quad (1.3)$$

$$\text{maximize } i_{\pm}(A - Z) \quad \text{subject to } Z \in \mathcal{S}, \quad (1.4)$$

$$\text{minimize } i_{\pm}(A - Z) \quad \text{subject to } Z \in \mathcal{S}, \quad (1.5)$$

respectively.

The rank and inertia of a Hermitian matrix are two generic concepts in matrix theory for describing the dimension of the row or column vector space and the sign distribution of the eigenvalues of the matrix, which are well understood and are easy to compute by the well-known elementary or congruent matrix operations. These two quantities play an essential role in characterizing relations between two matrices and algebraic properties of matrices. These two integer indices occur only in finite-dimensional algebras and are not replaceable and cannot be approximated by other continuous quantities. Because the rank and inertia of a matrix are always finite nonnegative integers less than or equal to the dimensions of the matrix, it is not hard to give upper and lower bounds for ranks and inertias of matrices, and the global maximal and minimal values of the integer-valued objective functions always exist, no matter what the decision domain \mathcal{S} is given. Also, due to the integer property of rank and inertia, inexact or approximate values of maximal and minimal ranks and inertias are less valuable, so that no approximation methods are allowed to use when finding the maximal and minimal possible ranks and inertias of a matrix-valued function. This fact means that solving methods of matrix rank and inertia optimization problems are not consistent with any of the ordinary continuous and discrete problems in optimization theory. It has been known that matrix rank optimization problems are NP-hard in general due to the discontinuity and combinatorial nature of rank of a matrix and the algebraic structure of \mathcal{S} . However, it is really lucky that we can establish analytical formulas for calculating the extremal ranks of matrix-valued functions for some special feasible matrix sets \mathcal{S} by using various expansion formulas for ranks and inertias of matrices and some tricky matrix operations.

Because the rank of a matrix can only take finite integers between 0 and the dimensions of the matrix, it is really expected to establish certain analytical formulas for calculating the maximal and minimal ranks for curiosity. In recent years, maximization and minimization problems on ranks and inertias of matrices attract much attention from both theoretical and practical points of view. In this paper, we assume that $A_i \in \mathbb{C}_{\text{H}}^{m_i}$ and $B_i \in \mathbb{C}^{m_i \times n}$, $i = 1, 2$ are given matrices, and the feasible matrix set \mathcal{S} is

$$\mathcal{S}_1 = \{Z = B_1XC_1 \mid X \in \mathbb{C}_{\text{H}}^n, B_2XB_2^* \succcurlyeq A_2\} \text{ or } \mathcal{S}_2 = \{Z = B_1XC_1 \mid X \in \mathbb{C}_{\text{H}}^n, B_2XB_2^* \preccurlyeq A_2\}. \quad (1.6)$$

Then, the difference $A_1 - Z$ can be written as the following linear matrix-valued function

$$\phi(X) = A_1 - B_1XC_1. \quad (1.7)$$

The LMIs in (1.6), the simplest cases of all LMIs, could be regarded as extensions of the usual inequalities $bx \geq a$ and $bx \leq a$ for real numbers.

Under such a formulation, this paper aims at solving the following inequality-constrained matrix optimization problems:

Problem 1.1 For the function in (1.7) and the feasible matrix sets in (1.6), establish explicit formulas for calculating the following extremal ranks and inertias

$$\max r(A_1 - B_1XB_1^*) \quad \text{s.t. } X \in \mathcal{S}_i, \quad i = 1, 2, \quad (1.8)$$

$$\min r(A_1 - B_1XB_1^*) \quad \text{s.t. } X \in \mathcal{S}_i, \quad i = 1, 2, \quad (1.9)$$

$$\max i_{\pm}(A_1 - B_1XB_1^*) \quad \text{s.t. } X \in \mathcal{S}_i, \quad i = 1, 2, \quad (1.10)$$

$$\min i_{\pm}(A_1 - B_1XB_1^*) \quad \text{s.t. } X \in \mathcal{S}_i, \quad i = 1, 2, \quad (1.11)$$

respectively.

Problem 1.2 Establish necessary sufficient conditions for the following two linear matrix inequalities (LMIs)

$$B_1XB_1^* \succcurlyeq A_1 \text{ and } B_2XB_2^* \succcurlyeq A_2 \quad (B_1XB_1^* \preccurlyeq A_1 \text{ and } B_2XB_2^* \preccurlyeq A_2) \quad (1.12)$$

to have a common Hermitian solution and give their common solutions.

Problem 1.3 For $\phi(X)$ in (1.7), establish necessary and sufficient conditions for the existence of $\hat{X}, \tilde{X} \in \mathbb{C}_{\text{H}}^n$ such that

$$\phi(\hat{X}) \preccurlyeq \phi(X) \preccurlyeq \phi(\tilde{X}) \quad \text{for all } B_2XB_2^* \succcurlyeq A_2 \text{ and } X \in \mathbb{C}_{\text{H}}^n, \quad (1.13)$$

$$\phi(\hat{X}) \preccurlyeq \phi(X) \preccurlyeq \phi(\tilde{X}) \quad \text{for all } B_2XB_2^* \preccurlyeq A_2 \text{ and } X \in \mathbb{C}_{\text{H}}^n \quad (1.14)$$

hold, respectively, and find analytical expressions of \hat{X} and \tilde{X} .

The matrix function $\phi(X) = A - BXB^*$, as one of the simplest cases among all matrix maps with symmetric patterns, attracted much attention in the recent decade, and many problems on $\phi(X)$, were considered in the literature. Some recent work on the matrix function is summarized below:

- (i) Expansion formulas for calculating the (global extremal) rank and inertia of $\phi(X)$ when X running over \mathbb{C}_H^n , [10, 16, 26].
- (ii) Nonsingularity, positive definiteness, rank and inertia invariance, etc., of $\phi(X)$, [16, 26].
- (iii) Canonical forms of $\phi(X)$ under generalized singular value decompositions and their algebraic properties, [10].
- (iv) Solutions and least-squares solutions of the matrix equation $\phi(X) = 0$ and their algebraic properties, [8, 10, 11, 17, 21, 24].
- (v) Minimization of $\text{tr}[\phi(X)\phi^*(X)]$ s.t. $r[\phi(X)] = \min$, [24].
- (vi) Solutions of the matrix inequalities $\phi(X) \succ (\succcurlyeq, \prec, \preccurlyeq) 0$ and their properties, [16, 20].
- (vii) Formulas for calculating the extremal rank and inertia of $\phi(X)$ under the restrictions $r(X) \leq k$ and/or $\pm X \succcurlyeq 0$, [16, 19, 20, 26].
- (viii) Formulas for calculating the extremal rank and inertia of $\phi(X)$ subject to the Hermitian solution of a consistent matrix equation $CXC^* = D$, [9].
- (ix) Formulas for calculating the extremal rank and inertia of the $A + BC^-B^*$, where C^- is a Hermitian generalized inverse of a Hermitian matrix C , [9, 22].

Mappings between matrix spaces with symmetric patterns can be constructed arbitrarily, but the linear function in (1.7) is the simplest cases among all matrix maps with symmetric patterns. The linear matrix inequality in (1.7) and its variations are usually taken as global convex constraints to unknown matrices and vectors in mathematical programming and optimization theory. Note that the commonly used definiteness matrices $X \succcurlyeq 0$ ($X \preccurlyeq 0$) is a special case of the inequality in (1.7). Thus, the inequality-constraints in (1.7) could be regarded as two extensions of definite matrix constraints arising in a number of optimization problems (see, e.g., [2, 4, 5, 14]). In fact, Problem 1.1 was proposed in the author's recent paper [20].

The above three problems are closely linked each other. Once analytical formulas for calculating the global maximal and minimal ranks and inertias in Problem 1.1 are obtained, we can easily use them to solve Problems 1.2 and 1.3.

The results in the following two lemmas are obvious or well-known (see also [16, 17] for their references), which we shall use in the latter part of this paper for solving the previous problems.

Lemma 1.4 *Let \mathcal{S} be a set consisting of matrices over $\mathbb{C}^{m \times n}$, and let \mathcal{H} be a set consisting of Hermitian matrices over \mathbb{C}_H^m . Then, the following hold.*

- (a) *Under $m = n$, \mathcal{S} has a nonsingular matrix if and only if $\max_{X \in \mathcal{S}} r(X) = m$.*
- (b) *Under $m = n$, all $X \in \mathcal{S}$ are nonsingular if and only if $\min_{X \in \mathcal{S}} r(X) = m$.*
- (c) *$0 \in \mathcal{S}$ if and only if $\min_{X \in \mathcal{S}} r(X) = 0$.*
- (d) *$\mathcal{S} = \{0\}$ if and only if $\max_{X \in \mathcal{S}} r(X) = 0$.*
- (e) *All $X \in \mathcal{S}$ have the same rank if and only if $\max_{X \in \mathcal{S}} r(X) = \min_{X \in \mathcal{S}} r(X)$.*
- (f) *\mathcal{H} has a matrix $X \succ 0$ ($X \prec 0$) if and only if $\max_{X \in \mathcal{H}} i_+(X) = m$ ($\max_{X \in \mathcal{H}} i_-(X) = m$).*
- (g) *All $X \in \mathcal{H}$ satisfy $X \succ 0$ ($X \prec 0$), namely, \mathcal{H} is a subset of the cone of positive definite matrices (negative definite matrices), if and only if $\min_{X \in \mathcal{H}} i_+(X) = m$ ($\min_{X \in \mathcal{H}} i_-(X) = m$).*
- (h) *\mathcal{H} has a matrix $X \succcurlyeq 0$ ($X \preccurlyeq 0$) if and only if $\min_{X \in \mathcal{H}} i_-(X) = 0$ ($\min_{X \in \mathcal{H}} i_+(X) = 0$).*
- (i) *All $X \in \mathcal{H}$ satisfy $X \succcurlyeq 0$ ($X \preccurlyeq 0$), namely, \mathcal{H} is a subset of the cone of positive semi-definite matrices (nonpositive definite matrices), if and only if $\max_{X \in \mathcal{H}} i_-(X) = 0$ ($\max_{X \in \mathcal{H}} i_+(X) = 0$).*
- (j) *All $X \in \mathcal{H}$ have the same positive index of inertia if and only if $\max_{X \in \mathcal{H}} i_+(X) = \min_{X \in \mathcal{H}} i_+(X)$.*
- (k) *All $X \in \mathcal{H}$ have the same negative index of inertia if and only if $\max_{X \in \mathcal{H}} i_-(X) = \min_{X \in \mathcal{H}} i_-(X)$.*

The question of whether a given function (matrix map), is positive or nonnegative (definite or semi-definite) everywhere is ubiquitous in mathematics and applications. Lemma 1.4(f)–(i) show that if some explicit formulas for calculating the global maximal and minimal inertias of a given Hermitian matrix map are established, we can use them, as demonstrated in Sections 2–5 below, to derive necessary and sufficient conditions for the Hermitian matrix map to be definite or semi-definite.

Lemma 1.5 Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}_H^n$, $Q \in \mathbb{C}^{m \times n}$, and assume that $P \in \mathbb{C}^{m \times m}$ is nonsingular. Then,

$$i_{\pm}(PAP^*) = i_{\pm}(A), \quad (1.15)$$

$$i_{\pm}(\lambda A) = \begin{cases} i_{\pm}(A) & \text{if } \lambda > 0 \\ i_{\mp}(A) & \text{if } \lambda < 0 \end{cases}, \quad (1.16)$$

$$i_{\pm} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_{\pm}(A) + i_{\pm}(B), \quad (1.17)$$

$$i_+ \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = i_- \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q). \quad (1.18)$$

Lemma 1.6 ([12]) Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$ and $C \in \mathbb{C}^{l \times n}$. Then, the following rank expansion formulas hold

$$r[A, B] = r(A) + r(E_A B) = r(B) + r(E_B A), \quad (1.19)$$

$$r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AC). \quad (1.20)$$

Lemma 1.7 ([16]) Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m \times n}$, $D \in \mathbb{C}_H^n$, and let

$$M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Then, the following expansion formulas hold

$$i_{\pm}(M_1) = r(B) + i_{\pm}(E_B A E_B), \quad r(M_1) = 2r(B) + r(E_B A E_B), \quad (1.21)$$

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm} \begin{bmatrix} 0 & E_A B \\ B^* E_A & D - B^* A^{\dagger} B \end{bmatrix}, \quad r(M_2) = r(A) + r \begin{bmatrix} 0 & E_A B \\ B^* E_A & D - B^* A^{\dagger} B \end{bmatrix}. \quad (1.22)$$

In particular, the following hold.

(a) If $A \succcurlyeq 0$, then

$$i_+(M_1) = r[A, B], \quad i_-(M_1) = r(B), \quad r(M_1) = r[A, B] + r(B). \quad (1.23)$$

(b) If $A \preccurlyeq 0$, then

$$i_+(M_1) = r(B), \quad i_-(M_1) = r[A, B], \quad r(M_1) = r[A, B] + r(B). \quad (1.24)$$

(c) If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}(D - B^* A^{\dagger} B), \quad r(M_2) = r(A) + r(D - B^* A^{\dagger} B). \quad (1.25)$$

(d) $i_{\pm}(M_2) \geq i_{\pm}(A) + i_{\pm}(D - B^* A^{\dagger} B) \geq i_{\pm}(A)$.

(e) $i_{\pm}(M_1) = m \Leftrightarrow i_{\mp}(E_B A E_B) = 0$ and $r(E_B A E_B) = r(E_B)$.

(f) $i_+(M_2) = i_+(A) \Leftrightarrow \mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $D - B^* A^{\dagger} B \preccurlyeq 0$.

(g) $i_-(M_2) = i_-(A) \Leftrightarrow \mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $D - B^* A^{\dagger} B \succcurlyeq 0$.

(h) $M_2 \succcurlyeq 0 \Leftrightarrow A \succcurlyeq 0$, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $D - B^* A^{\dagger} B \succcurlyeq 0 \Leftrightarrow D \succcurlyeq 0$, $\mathcal{R}(B^*) \subseteq \mathcal{R}(D)$ and $A - BD^{\dagger}B^* \succcurlyeq 0$.

(i) $M_2 \succ 0 \Leftrightarrow A \succ 0$ and $D - B^* A^{-1} B \succ 0 \Leftrightarrow D \succ 0$ and $A - BD^{-1}B^* \succ 0$.

Some useful expansion formulas derived from (1.21) and (1.22) are

$$i_{\pm}(D - B^* A^{\dagger} B) = i_{\pm} \begin{bmatrix} A^* A A^* & A^* B \\ B^* A^* & D \end{bmatrix} - i_{\pm}(A), \quad (1.26)$$

$$i_{\pm}(D - B^* A^{\dagger} B) = i_{\pm} \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} - i_{\pm}(A) \quad \text{if } \mathcal{R}(B) \subseteq \mathcal{R}(A), \quad (1.27)$$

$$i_{\pm} \begin{bmatrix} A & B F_P \\ F_P B^* & 0 \end{bmatrix} = i_{\pm} \begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - r(P). \quad (1.28)$$

We shall use them to simplify the inertias of block Hermitian matrices involving Moore–Penrose inverses of matrices.

Lemma 1.8 ([15]) Let $A \in \mathbb{C}_H^m$ and $B \in \mathbb{C}^{m \times n}$. Then, the following expansion formula

$$\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}^\dagger = \begin{bmatrix} (E_B A E_B)^\dagger & (B^\dagger)^* - (E_B A E_B)^\dagger E_B A (B^\dagger)^* \\ B^\dagger - B^\dagger A E_B (E_B A E_B)^\dagger & -B^\dagger A (B^\dagger)^* + B^\dagger A E_B (E_B A E_B)^\dagger E_B A (B^\dagger)^* \end{bmatrix}$$

holds if and only if

$$r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} = r[A, B] + r(B),$$

or equivalently, $r(E_B A E_B) = r(E_B A)$.

Solving matrix equations is one of the key problems of matrix computation. Many techniques were proposed and developed in studying consistency and solutions of various matrix equations. In this paper, we need the following results on solvability conditions and general solutions of two simple linear matrix equations.

Lemma 1.9 ([6]) Let $A, B \in \mathbb{C}^{m \times n}$ be given. Then, the following hold.

- (a) The matrix equation $AX = B$ has a solution $X \in \mathbb{C}_H^n$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $AB^* = BA^*$. In this case, the general Hermitian solution can be written in the following parametric form

$$X = A^\dagger B + (A^\dagger B)^* - A^\dagger B A^\dagger A + F_A U F_A, \quad (1.29)$$

where $U \in \mathbb{C}_H^n$ is arbitrary.

- (b) $AX = B$ has a solution $0 \preceq X \in \mathbb{C}_H^n$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, $AB^* \succcurlyeq 0$ and $r(AB^*) = r(B)$. In this case, the general solution $0 \preceq X \in \mathbb{C}_H^n$ can be written as

$$X = B^*(AB^*)^\dagger B + F_A U U^* F_A, \quad (1.30)$$

where $U \in \mathbb{C}^{n \times n}$ is arbitrary.

Lemma 1.10 Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}_H^m$ be given. Then, the following hold.

- (a) [3] The matrix equation

$$AXA^* = B \quad (1.31)$$

has a solution $X \in \mathbb{C}_H^n$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, or equivalently, $AA^\dagger B = B$.

- (b) [16] Under $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, the general Hermitian solution of (1.31) can be written in the following two forms

$$X = A^\dagger B (A^\dagger)^* + F_A V + V^* F_A, \quad (1.32)$$

respectively, where $V \in \mathbb{C}^{n \times n}$ is arbitrary.

- (c) [3, 6] The matrix equation

$$AXA^* = B \quad (1.33)$$

has a solution $0 \preceq X \in \mathbb{C}_H^n$ if and only if $B \succcurlyeq 0$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. In this case, the general positive semi-definite solution of (1.33) can be written in the following parametric form

$$X = A^- B (A^-)^* + F_A U U^* F_A = (A^\dagger + F_A V) B (A^\dagger + F_A V)^* + F_A U U^* F_A, \quad (1.34)$$

where A^- is an arbitrary g -inverse of A , and $V \in \mathbb{C}^{n \times m}$ and $U \in \mathbb{C}^{n \times n}$ are arbitrary.

In order to simplify various matrix-valued function involving generalized inverse of matrices and arbitrary matrices, we need the following results on ranks of matrices.

Lemma 1.11 ([9, 17]) Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m \times n}$ and $C \in \mathbb{C}^{p \times m}$ be given. Then, the global maximal and minimal ranks and inertias of $A - BXC - (BXC)^*$ are given by

$$\max_{X \in \mathbb{C}^{n \times p}} r[A - BXC - (BXC)^*] = \min \left\{ r[A, B, C^*], \quad r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad r \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \quad (1.35)$$

$$\min_{X \in \mathbb{C}^{n \times p}} r[A - BXC - (BXC)^*] = 2r[A, B, C^*] + \max\{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\}, \quad (1.36)$$

$$\max_{X \in \mathbb{C}^{n \times p}} i_\pm[A - BXC - (BXC)^*] = \min \left\{ i_\pm \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad i_\pm \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \quad (1.37)$$

$$\min_{X \in \mathbb{C}^{n \times p}} i_\pm[A - BXC - (BXC)^*] = r[A, B, C^*] + \max\{s_\pm, t_\pm\}, \quad (1.38)$$

where

$$s_{\pm} = i_{\pm} \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r \begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix}, \quad t_{\pm} = i_{\pm} \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - r \begin{bmatrix} A & B & C^* \\ C & 0 & 0 \end{bmatrix}.$$

In particular, if $\mathcal{R}(B) \subseteq \mathcal{R}(C^*)$, then

$$\max_{X \in \mathbb{C}^{n \times p}} r[A - BXC - (BXC)^*] = \min \left\{ r[A, C^*], \quad r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \right\}, \quad (1.39)$$

$$\min_{X \in \mathbb{C}^{n \times p}} r[A - BXC - (BXC)^*] = 2r[A, C^*] + r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \quad (1.40)$$

$$\max_{X \in \mathbb{C}^{n \times p}} i_{\pm}[A - BXC - (BXC)^*] = i_{\pm} \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad (1.41)$$

$$\min_{X \in \mathbb{C}^{n \times p}} i_{\pm}[A - BXC - (BXC)^*] = r[A, C^*] + i_{\pm} \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \quad (1.42)$$

and

$$\max_{X \in \mathbb{C}^{n \times m}} r[A - BX - (BX)^*] = \min \left\{ m, \quad r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \right\}, \quad (1.43)$$

$$\min_{X \in \mathbb{C}^{n \times m}} r[A - BX - (BX)^*] = r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r(B), \quad (1.44)$$

$$\max_{X \in \mathbb{C}^{n \times m}} i_{\pm}[A - BX - (BX)^*] = i_{\pm} \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad (1.45)$$

$$\min_{X \in \mathbb{C}^{n \times m}} i_{\pm}[A - BX - (BX)^*] = i_{\pm} \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r(B). \quad (1.46)$$

This paper is organized as follows. In Section 2, we derive general Hermitian solution of the LMI in (1.7) by using generalized inverses of matrices, Lemmas 1.9 and 1.10, and present some algebraic properties of the Hermitian solutions. In Sections 3–5, we derive explicit solutions to Problems 1.1–1.3, and present various consequences of the rank and inertia formulas obtained. In Section 6, we calculate the global maximal and minimal ranks and inertias of the Hermitian solution of $BXB^* \succcurlyeq A$, as well as the global maximal and minimal ranks and inertias of the submatrices in a Hermitian solution of $BXB^* \succcurlyeq A$.

2 General Hermitian solutions of the LMIs $BXB^* \succcurlyeq (\succ, \preccurlyeq, \prec) A$ and their properties

Concerning the global maximal and minimal ranks and inertias of (1.7), we have the following known result.

Lemma 2.1 ([16, 26]) *Let $A \in \mathbb{C}_H^m$ and $B \in \mathbb{C}^{m \times n}$ be given, and define $M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}$. Then, the global maximal and minimal rank and inertias of $A - BXB^*$ are given by*

$$\max_{X \in \mathbb{C}_H^n} r(A - BXB^*) = r[A, B], \quad (2.1)$$

$$\min_{X \in \mathbb{C}_H^n} r(A - BXB^*) = 2r[A, B] - r(M), \quad (2.2)$$

$$\max_{X \in \mathbb{C}_H^n} i_{\pm}(A - BXB^*) = i_{\pm}(M), \quad (2.3)$$

$$\min_{X \in \mathbb{C}_H^n} i_{\pm}(A - BXB^*) = r[A, B] - i_{\mp}(M). \quad (2.4)$$

We next solve the two inequalities in (1.6) and give their general Hermitian solutions by using Lemmas 1.9 and 1.10, some partial conclusions were given in [25].

Theorem 2.2 *Let $A \in \mathbb{C}_H^m$ and $B \in \mathbb{C}^{m \times n}$ be given. Then, the following hold.*

(a) *There exists an $X \in \mathbb{C}_H^n$ such that*

$$BXB^* \succcurlyeq A \quad (2.5)$$

if and only if

$$E_B A E_B \preccurlyeq 0 \quad \text{and} \quad r(E_B A E_B) = r(E_B A), \quad (2.6)$$

or equivalently,

$$i_+(M) = r(B) \quad \text{and} \quad i_-(M) = r[A, B]. \quad (2.7)$$

In this case, the general Hermitian solution of (2.5) can be written as

$$X = B^\dagger A(B^\dagger)^* - B^\dagger A E_B (E_B A E_B)^\dagger E_B A (B^\dagger)^* + U U^* + F_B V + V^* F_B, \quad (2.8)$$

$$B X B^* = A - A E_B (E_B A E_B)^\dagger E_B A + B U U^* B^*, \quad (2.9)$$

$$A - B X B^* = A E_B (E_B A E_B)^\dagger E_B A - B U U^* B^*, \quad (2.10)$$

where $U, V \in \mathbb{C}^{n \times n}$ are arbitrary.

(b) There exists an $X \in \mathbb{C}_H^n$ such that

$$B X B^* \succ A \quad (2.11)$$

if and only if

$$E_B A E_B \preceq 0 \quad \text{and} \quad r(E_B A E_B) = r(E_B) \quad (2.12)$$

hold. In this case, the general Hermitian solution of (2.11) can be written as (2.8), in which U is a matrix such that $r[A E_B (E_B A E_B)^\dagger E_B A - B U U^* B^*] = m$, and $V \in \mathbb{C}^{n \times n}$ is arbitrary.

In particular, the following hold.

(c) If $B X B^* = A$ is consistent, then the general Hermitian solution of $B X B^* \succ A$ can be written as

$$X = B^\dagger A(B^\dagger)^* + U U^* + F_B V + V^* F_B, \quad (2.13)$$

where $U, V \in \mathbb{C}^{n \times n}$ are arbitrary.

(d) If $B X B^* = A$ is consistent, then $B X B^* \succ A$ has a Hermitian solution if and only if $r(B) = m$, in which case, the general Hermitian solution of the LMI can be written as

$$X = B^\dagger A(B^\dagger)^* + U + F_B V + V^* F_B, \quad (2.14)$$

where $U \in \mathbb{C}_H^n$ is arbitrary matrix such that $B U B^* \succ 0$, and $V \in \mathbb{C}^{n \times n}$ is arbitrary.

Proof. It is obvious that (2.5) is equivalent to

$$B X B^* = A + Y Y^* \quad (2.15)$$

for some matrix Y . In other words, (2.5) can be relaxed to a matrix equation with two unknown matrices. We obtain from Lemma 1.10(a) that (2.15) is solvable for $X \in \mathbb{C}_H^n$ if and only if $E_B(A + Y Y^*) = 0$, that is,

$$E_B Y Y^* = -E_B A. \quad (2.16)$$

From Lemma 1.9(b), (2.16) is solvable for $Y Y^*$ if and only if $E_B A E_B \preceq 0$ and $r(E_B A E_B) = r(E_B A)$, establishing (2.6), which is further equivalent to (2.7) by (1.19) and (1.21). In this case, the general positive semi-definite solution of (2.16) can be written as

$$Y Y^* = -A E_B (E_B A E_B)^\dagger E_B A + B B^\dagger U U^* B B^\dagger, \quad (2.17)$$

where $U \in \mathbb{C}^{m \times m}$ is arbitrary. Substituting the $Y Y^*$ into (2.15) gives

$$B X B^* = A - A E_B (E_B A E_B)^\dagger E_B A + B B^\dagger U U^* B B^\dagger. \quad (2.18)$$

By Lemma 1.10(b), the general Hermitian solution of (2.18) can be written as

$$X = B^\dagger A(B^\dagger)^* - B^\dagger A E_B (E_B A E_B)^\dagger E_B A (B^\dagger)^* + B^\dagger U U^* (B^\dagger)^* + F_B V + V^* F_B, \quad (2.19)$$

where $V \in \mathbb{C}^{n \times n}$ is arbitrary. Replacing the matrix $B^\dagger U U^* (B^\dagger)^*$ in (2.19) with $U U^*$ yields (2.8), which is also the general Hermitian solution of (2.5).

Substituting (2.8) into $A - B X B^*$ gives

$$\begin{aligned} A - B X B^* &= A - B B^\dagger A B B^\dagger + B B^\dagger A E_B (E_B A E_B)^\dagger E_B A B B^\dagger - B U U^* B^* \\ &= A E_B (E_B A E_B)^\dagger E_B A - B U U^* B^*, \end{aligned} \quad (2.20)$$

Note that $A E_B (E_B A E_B)^\dagger E_B A \preceq 0$. Then, we have

$$\begin{aligned} i_-[A E_B (E_B A E_B)^\dagger E_B A - B U U^* B^*] &= r[A E_B (E_B A E_B)^\dagger E_B A - B U U^* B^*] \\ &= r[A E_B (E_B A E_B)^\dagger E_B A, B U U^* B^*] = r[A E_B, B U]. \end{aligned}$$

In consequence,

$$\max_{U \succeq 0} i_-[A E_B (E_B A E_B)^\dagger E_B A - B U U^* B^*] = \max_U r[A E_B, B U] = r[A E_B, B] = r(E_B A E_B) + r(B),$$

so that (2.11) holds if and only if $r(E_B A E_B) + r(B) = m$, establishing (b). If $B X B^* = A$ is consistent, then $E_B A = 0$. In this case, (a) and (b) reduce to (c) and (d). \square

Replacing A with $-A$, and X with $-X$ in Theorem 2.2 leads to the following consequence.

Corollary 2.3 Let $A \in \mathbb{C}_H^m$ and $B \in \mathbb{C}^{m \times n}$ be given. Then, the following hold.

(a) There exists an $X \in \mathbb{C}_H^n$ such that

$$BXB^* \preceq A \quad (2.21)$$

if and only if

$$E_B A E_B \succcurlyeq 0 \quad \text{and} \quad r(E_B A E_B) = r(E_B A), \quad (2.22)$$

or equivalently,

$$i_+(M) = r[A, B] \quad \text{and} \quad i_-(M) = r(B). \quad (2.23)$$

In this case, the general Hermitian solution of (2.21) can be written in the following parametric form

$$X = B^\dagger A (B^\dagger)^* - B^\dagger A E_B (E_B A E_B)^\dagger E_B A (B^\dagger)^* - U U^* + F_B V + V^* F_B, \quad (2.24)$$

$$B X B^* = A - A E_B (E_B A E_B)^\dagger E_B A - B U U^* B^*, \quad (2.25)$$

$$A - B X B^* = A E_B (E_B A E_B)^\dagger E_B A + B U U^* B^*, \quad (2.26)$$

where $U, V \in \mathbb{C}^{n \times n}$ are arbitrary.

(b) There exists an $X \in \mathbb{C}_H^n$ such that

$$B X B^* \prec A \quad (2.27)$$

if and only if

$$E_B A E_B \succcurlyeq 0 \quad \text{and} \quad r(E_B A E_B) = r(E_B) \quad (2.28)$$

hold. In this case, the general Hermitian solution of (2.27) can be written as (2.24), in which U is a matrix such that $r[A E_B (E_B A E_B)^\dagger E_B A - B U U^* B^*] = m$, and $V \in \mathbb{C}^{n \times n}$ is arbitrary.

In particular,

(c) If $B X B^* = A$ is consistent, then the general Hermitian solution of $B X B^* \preceq A$ can be written as

$$X = B^\dagger A (B^\dagger)^* - U U^* + F_B V + V^* F_B, \quad (2.29)$$

where $U, V \in \mathbb{C}^{n \times n}$ are arbitrary.

(d) If $B X B^* = A$ is consistent, then of $B X B^* \prec A$ has a Hermitian solution if and only if $r(B) = m$, in which case, the general Hermitian solution of the LMI can be written as

$$X = B^\dagger A (B^\dagger)^* - U + F_B V + V^* F_B, \quad (2.30)$$

where $U \in \mathbb{C}_H^n$ is arbitrary matrix such that $B U B^* \succ 0$, and $V \in \mathbb{C}^{n \times n}$ is arbitrary.

We next establish some algebraic properties of the fixed part in (2.8) and (2.24).

Corollary 2.4 Let $A \in \mathbb{C}_H^m$ and $B \in \mathbb{C}^{m \times n}$ be given, and let

$$\widehat{X} = B^\dagger A (B^\dagger)^* - B^\dagger A E_B (E_B A E_B)^\dagger E_B A (B^\dagger)^*. \quad (2.31)$$

Then, the following hold.

(a) Under the condition in (2.6),

(i) \widehat{X} is a Hermitian solution of $B X B^* \succcurlyeq A$.

(ii) \widehat{X} can be written as $\widehat{X} = [0, I_n] \begin{bmatrix} -A & B \\ B^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ I_n \end{bmatrix}$.

(iii) \widehat{X} satisfies the following equalities

$$i_+(\widehat{X}) = i_+(B \widehat{X} B^*) = i_+(A), \quad (2.32)$$

$$i_-(\widehat{X}) = i_-(B \widehat{X} B^*) = i_-(A) + r(B) - r[A, B], \quad (2.33)$$

$$r(\widehat{X}) = r(B \widehat{X} B^*) = r(A) + r(B) - r[A, B], \quad (2.34)$$

$$i_-(A - B \widehat{X} B^*) = r(A - B \widehat{X} B^*) = r(A) - r(B \widehat{X} B^*) = r[A, B] - r(B), \quad (2.35)$$

$$\max_{\succcurlyeq} \{A - B X B^* \mid B X B^* \succcurlyeq A \text{ and } X \in \mathbb{C}_H^n\} = A - B \widehat{X} B^*. \quad (2.36)$$

(b) Under the condition in (2.22),

(i) \widehat{X} is a Hermitian solution of $B X B^* \preceq A$.

$$(ii) \quad \widehat{X} \text{ can be written as } \widehat{X} = [0, I_n] \begin{bmatrix} -A & B \\ B^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ I_n \end{bmatrix}.$$

(iii) \widehat{X} satisfies the following equalities

$$i_+(\widehat{X}) = i_+(B\widehat{X}B^*) = i_+(A) + r(B) - r[A, B], \quad (2.37)$$

$$i_-(\widehat{X}) = i_-(B\widehat{X}B^*) = i_-(A), \quad (2.38)$$

$$r(\widehat{X}) = r(B\widehat{X}B^*) = r(A) + r(B) - r[A, B], \quad (2.39)$$

$$i_+(A - B\widehat{X}B^*) = r(A - B\widehat{X}B^*) = r(A) - r(B\widehat{X}B^*) = r[A, B] - r(B), \quad (2.40)$$

$$\min_{\succcurlyeq} \{A - BXB^* \mid BXB^* \preccurlyeq A \text{ and } X \in \mathbb{C}_H^n\} = A - B\widehat{X}B^*. \quad (2.41)$$

Proof. Under the condition in (2.5), comparing (2.31) with Lemma 1.5 leads to (ii) of (a).

Applying (1.27) to (2.31) and simplifying by congruence matrix operations, we obtain

$$\begin{aligned} i_\pm(\widehat{X}) &= i_\pm[B^\dagger A(B^\dagger)^* - B^\dagger A E_B (E_B A E_B)^\dagger E_B A (B^\dagger)^*] \\ &= i_\pm \begin{bmatrix} E_B A E_B & E_B A (B^\dagger)^* \\ B^\dagger A E_B & B^\dagger A (B^\dagger)^* \end{bmatrix} - i_\pm(E_B A E_B) \\ &= i_\pm \begin{bmatrix} A & A(B^\dagger)^* \\ B^\dagger A & B^\dagger A (B^\dagger)^* \end{bmatrix} - i_\pm(E_B A E_B) = i_\pm(A) - i_\pm(E_B A E_B). \end{aligned} \quad (2.42)$$

In consequence,

$$i_+(\widehat{X}) = i_+(A), \quad i_-(\widehat{X}) = i_-(A) - i_-(E_B A E_B) = i_-(A) - r(E_B A) = i_-(A) + r(B) - r[A, B],$$

establishing (iii) of (a).

Under the condition in (2.6), applying (1.27) and simplifying by congruence matrix operations, we obtain

$$\begin{aligned} i_\pm(A - B\widehat{X}B^*) &= i_\pm[A - BB^\dagger ABB^\dagger + BB^\dagger A E_B (E_B A E_B)^\dagger E_B ABB^\dagger] \\ &= i_\pm \begin{bmatrix} -E_B A E_B & E_B ABB^\dagger \\ BB^\dagger A E_B & A - BB^\dagger ABB^\dagger \end{bmatrix} - i_\mp(E_B A E_B) \\ &= i_\pm \begin{bmatrix} -E_B A E_B & E_B A \\ A E_B & 0 \end{bmatrix} - i_\mp(E_B A E_B) \\ &= r(E_B A) - i_\mp(E_B A E_B). \end{aligned} \quad (2.43)$$

In consequence,

$$\begin{aligned} i_+(A - B\widehat{X}B^*) &= r(E_B A) - i_-(E_B A E_B) = r(E_B A) - r(E_B A E_B) = 0, \\ i_-(A - B\widehat{X}B^*) &= r(E_B A) - i_+(E_B A E_B) = r[A, B] - r(B), \end{aligned}$$

establishing (iv) of (a).

Substituting (2.8) into $A - BXB^*$ gives

$$A - BXB^* = A - B\widehat{X}B^* - BUU^*B^* \preccurlyeq A - B\widehat{X}B^* \quad (2.44)$$

for any $U \in \mathbb{C}^{n \times n}$, which implies (v) of (a). Result (b) can be shown similarly. \square

Corollary 2.5 Let $A \in \mathbb{C}_H^m$ and $B \in \mathbb{C}^{m \times n}$ be given.

(a) Assume that (2.5) has a solution, and define

$$\mathcal{S}_1 = \{X \in \mathbb{C}_H^n \mid BXB^* \succcurlyeq A\}. \quad (2.45)$$

Then, the minimal matrices of BXB^* and $BXB^* - A$ subject to $X \in \mathcal{S}_1$ in the Löwner partial ordering are given by

$$\min_{\succcurlyeq} \{BXB^* \mid X \in \mathcal{S}_1\} = A - A E_B (E_B A E_B)^\dagger E_B A, \quad (2.46)$$

$$\min_{\succcurlyeq} \{BXB^* - A \mid X \in \mathcal{S}_1\} = -A E_B (E_B A E_B)^\dagger E_B A, \quad (2.47)$$

while the extremal ranks and inertias of BXB^* and $BXB^* - A$ subject to $X \in \mathcal{S}_1$ are given by

$$\max_{X \in \mathcal{S}_1} r(BXB^*) = \max_{X \in \mathcal{S}_1} i_+(BXB^*) = r(B), \quad (2.48)$$

$$\min_{X \in \mathcal{S}_1} r(BXB^*) = \min_{X \in \mathcal{S}_1} i_+(BXB^*) = i_+(A), \quad (2.49)$$

$$\max_{X \in \mathcal{S}_1} i_-(BXB^*) = r(B) + i_-(A) - r[A, B], \quad (2.50)$$

$$\min_{X \in \mathcal{S}_1} i_-(BXB^*) = 0, \quad (2.51)$$

$$\max_{X \in \mathcal{S}_1} r(BXB^* - A) = r[A, B], \quad (2.52)$$

$$\min_{X \in \mathcal{S}_1} r(BXB^* - A) = r[A, B] - r(B). \quad (2.53)$$

(b) Assume that (2.21) has a solution, and define

$$\mathcal{S}_2 = \{X \in \mathbb{C}_H^n \mid BXB^* \preceq A\}. \quad (2.54)$$

Then, the maximal matrices of BXB^* and $BXB^* - A$ subject to $X \in \mathcal{S}_2$ in the Löwner partial ordering are given by

$$\max_{\succcurlyeq} \{BXB^* \mid X \in \mathcal{S}_2\} = A - AE_B(E_BAE_B)^\dagger E_BA, \quad (2.55)$$

$$\max_{\succcurlyeq} \{BXB^* - A \mid X \in \mathcal{S}_2\} = -AE_B(E_BAE_B)^\dagger E_BA, \quad (2.56)$$

while the extremal ranks and inertias of BXB^* and $BXB^* - A$ subject to $X \in \mathcal{S}_2$ are given by

$$\max_{X \in \mathcal{S}_2} r(BXB^*) = \max_{X \in \mathcal{S}_2} i_-(BXB^*) = r(B), \quad (2.57)$$

$$\min_{X \in \mathcal{S}_2} r(BXB^*) = \min_{X \in \mathcal{S}_2} i_-(BXB^*) = i_-(A), \quad (2.58)$$

$$\max_{X \in \mathcal{S}_2} i_+(BXB^*) = r(B) + i_+(A) - r[A, B], \quad (2.59)$$

$$\min_{X \in \mathcal{S}_2} i_+(BXB^*) = 0, \quad (2.60)$$

$$\max_{X \in \mathcal{S}_2} r(A - BXB^*) = r[A, B], \quad (2.61)$$

$$\min_{X \in \mathcal{S}_2} r(A - BXB^*) = r[A, B] - r(B). \quad (2.62)$$

Proof. It can be seen from (2.9) that

$$BXB^* \succcurlyeq A - AE_B(E_BAE_B)^\dagger E_BA, \quad BXB^* - A \succcurlyeq -AE_B(E_BAE_B)^\dagger E_BA \quad (2.63)$$

hold for any $U \in \mathbb{C}^{n \times n}$, which implies (2.46) and (2.47). Applying (3.13)–(3.18) to (2.9) and simplifying by congruence matrix operations, we obtain

$$\begin{aligned} \max_{X \in \mathcal{S}_1} r(BXB^*) &= \max_{U \in \mathbb{C}^{n \times n}} r(B\hat{X}B^* + BUU^*B^*) = r[B, B\hat{X}B^*] = r(B), \\ \min_{X \in \mathcal{S}_1} r(BXB^*) &= \min_{U \in \mathbb{C}^{n \times n}} r(B\hat{X}B^* + BUU^*B^*) \\ &= i_+(B\hat{X}B^*) + r[B, B\hat{X}B^*] - i_+\begin{bmatrix} B\hat{X}B^* & B \\ B^* & 0 \end{bmatrix} = i_+(B\hat{X}B^*) = i_+(A), \\ \max_{X \in \mathcal{S}_1} i_+(BXB^*) &= \max_{U \in \mathbb{C}^{n \times n}} i_+(B\hat{X}B^* + BUU^*B^*) = i_+\begin{bmatrix} B\hat{X}B^* & B \\ B^* & 0 \end{bmatrix} = r(B), \\ \min_{X \in \mathcal{S}_1} i_+(BXB^*) &= \min_{U \in \mathbb{C}^{n \times n}} i_+(B\hat{X}B^* + BUU^*B^*) = i_+(B\hat{X}B^*) = i_+(A), \\ \max_{X \in \mathcal{S}_1} i_-(BXB^*) &= \max_{U \in \mathbb{C}^{n \times n}} i_-(B\hat{X}B^* + BUU^*B^*) = i_-(B\hat{X}B^*) = r(B) + i_-(A) - r[B, A], \\ \min_{X \in \mathcal{S}_1} i_-(BXB^*) &= \min_{U \in \mathbb{C}^{n \times n}} i_-(B\hat{X}B^* + BUU^*B^*) = r[B, B\hat{X}B^*] - i_+\begin{bmatrix} B\hat{X}B^* & B \\ B^* & 0 \end{bmatrix} = 0, \end{aligned}$$

establishing (2.46)–(2.49). Applying (1.19), (1.20) to (2.9), we obtain

$$\begin{aligned} \max_{X \in \mathcal{S}_1} r(A - BXB^*) &= \max_{U \in \mathbb{C}^{n \times n}} r[AE_B, BU] = r[AE_B, B] = r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r(B) = r[A, B], \\ \min_{X \in \mathcal{S}_1} r(A - BXB^*) &= \min_{U \in \mathbb{C}^{n \times n}} r[AE_B, BU] = r(AE_B) = r[A, B] - r(B), \end{aligned}$$

establishing (2.52) and (2.53). Result (b) can be shown similarly. \square

Some direct consequences of Theorem 2.2 are given below.

Corollary 2.6 *Let $A \in \mathbb{C}_H^m$ and $B \in \mathbb{C}^{m \times n}$ be given. Then, the following hold.*

- (a) *There exists an $X \in \mathbb{C}_H^n$ such that $BXB^* \succcurlyeq A \succcurlyeq 0$ if and only if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, namely, the matrix equation $BY = A$ is consistent. In this case, the general Hermitian solution can be written as*

$$X = B^\dagger A (B^\dagger)^* + UU^* + F_B V + V^* F_B, \quad (2.64)$$

where $U, V \in \mathbb{C}^{n \times n}$ are arbitrary.

- (b) *There exists an $X \in \mathbb{C}_H^n$ such that $BXB^* \succcurlyeq A \succcurlyeq 0$ if and only if $r(B) = m$. In this case, the general Hermitian solution can be written as (2.64), in which $U, V \in \mathbb{C}^{n \times n}$ are arbitrary.*

- (c) *There exists an $X \in \mathbb{C}_H^n$ such that $BXB^* \preccurlyeq A \preccurlyeq 0$ if and only if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. namely, the matrix equation $BY = A$ is consistent. In this case, the general Hermitian solution can be written as*

$$X = B^\dagger A (B^\dagger)^* - UU^* + F_A V + V^* F_A, \quad (2.65)$$

where $U, V \in \mathbb{C}^{n \times n}$ are arbitrary.

- (d) *There exists an $X \in \mathbb{C}^{n \times n}$ such that $BXB^* \prec A \preccurlyeq 0$ if and only if $r(B) = m$. In this case, the general Hermitian solution can be written as (2.65), in which $U, V \in \mathbb{C}^{n \times n}$ are arbitrary.*

In particular,

- (e) *There exists a $0 \preccurlyeq X \in \mathbb{C}_H^n$ such that $BXB^* \succcurlyeq A \succcurlyeq 0$ if and only if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, namely, the matrix equation $BY = A$ is consistent. In this case, the general positive semi-definite solution can be written as*

$$X = B^\dagger A (B^\dagger)^* + UU^*, \quad (2.66)$$

where $U \in \mathbb{C}^{n \times n}$ is arbitrary.

- (f) *There exists a $0 \preccurlyeq X \in \mathbb{C}_H^n$ such that*

$$BXB^* \succcurlyeq A \succcurlyeq 0 \quad (2.67)$$

if and only if $r(B) = m$. In this case, the general general positive semi-definite solution of (2.67) can be written as (2.66), in which $U \in \mathbb{C}^{n \times n}$ is arbitrary.

Proof. Under the condition $A \succcurlyeq 0$, (2.6) is equivalent to $E_B A = 0$, i.e., $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. In this case, (2.8) reduces to (2.64). Also under the condition $A \succcurlyeq 0$, (2.12) is equivalent to $E_B = 0$ i.e., $BB^\dagger = I_m$, which is further equivalent to $r(B) = m$, as required for (b). Results (c) and (d) can be shown similarly. Results (e) and (f) follow from (a) and (b). \square

Note that the formulas in (2.8) and (2.24) are given in closed-form with two independent parametric matrices. Hence, it is easy to use the two formulas in the investigation of algebraic properties of the LMI in (1.7) and various problems related to the LMI. In Theorem 3.2 below, we shall give the global maximal and minimal ranks and inertias of the two Hermitian solutions in (2.8) and (2.24); in Section 3, we shall use (2.8) and (2.24) to solve the inequality-constrained rank and inertia optimization problems in (1.8)–(1.11).

3 Ranks and inertias of $A_1 - B_1 X B_1^*$ subject to $B_2 X B_2^* \succcurlyeq A_2$

Note that (2.8) is in fact a quadratic form, so that (1.7) is a quadratic form as well. To solve (1.8)–(1.11), we need the following known results.

Lemma 3.1 ([20, 23]) *Let $A \in \mathbb{C}_H^m$ and $B \in \mathbb{C}^{m \times n}$ be given, and let $M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}$. Then,*

$$\max_{X \in \mathbb{C}^{n \times k}} r(A + BXX^*B^*) = \min\{r[A, B], \quad k + r(A)\}, \quad (3.1)$$

$$\min_{X \in \mathbb{C}^{n \times k}} r(A + BXX^*B^*) = \max\{r(A) - k, \quad i_+(A) + r[A, B] - i_+(M)\}, \quad (3.2)$$

$$\max_{X \in \mathbb{C}^{n \times k}} i_+(A + BXX^*B^*) = \min\{i_+(M), \quad k + i_+(A)\}, \quad (3.3)$$

$$\min_{X \in \mathbb{C}^{n \times k}} i_+(A + BXX^*B^*) = i_+(A), \quad (3.4)$$

$$\max_{X \in \mathbb{C}^{n \times k}} i_-(A + BXX^*B^*) = i_-(A), \quad (3.5)$$

$$\min_{X \in \mathbb{C}^{n \times k}} i_-(A + BXX^*B^*) = \max\{i_-(A) - k, \quad r[A, B] - i_+(M)\}, \quad (3.6)$$

$$\max_{X \in \mathbb{C}^{n \times k}} r(A - BXX^*B^*) = \min\{r[A, B], \quad k + r(A)\}, \quad (3.7)$$

$$\min_{X \in \mathbb{C}^{n \times k}} r(A - BXX^*B^*) = \max\{r(A) - k, \quad i_-(A) + r[A, B] - i_-(M)\}, \quad (3.8)$$

$$\max_{X \in \mathbb{C}^{n \times k}} i_+(A - BXX^*B^*) = i_+(A), \quad (3.9)$$

$$\min_{X \in \mathbb{C}^{n \times k}} i_+(A - BXX^*B^*) = \max\{i_+(A) - k, \quad r[A, B] - i_-(M)\}, \quad (3.10)$$

$$\max_{X \in \mathbb{C}^{n \times k}} i_-(A - BXX^*B^*) = \min\{i_-(M), \quad k + i_-(A)\}, \quad (3.11)$$

$$\min_{X \in \mathbb{C}^{n \times k}} i_-(A - BXX^*B^*) = i_-(A). \quad (3.12)$$

In particular,

$$\max_{X \in \mathbb{C}^{n \times n}} r(A + BXX^*B^*) = r[A, B], \quad (3.13)$$

$$\min_{X \in \mathbb{C}^{n \times n}} r(A + BXX^*B^*) = i_+(A) + r[A, B] - i_+(M), \quad (3.14)$$

$$\max_{X \in \mathbb{C}^{n \times n}} i_+(A + BXX^*B^*) = i_+(M), \quad (3.15)$$

$$\min_{X \in \mathbb{C}^{n \times n}} i_+(A + BXX^*B^*) = i_+(A), \quad (3.16)$$

$$\max_{X \in \mathbb{C}^{n \times n}} i_-(A + BXX^*B^*) = i_-(A), \quad (3.17)$$

$$\min_{X \in \mathbb{C}^{n \times n}} i_-(A + BXX^*B^*) = r[A, B] - i_+(M), \quad (3.18)$$

$$\max_{X \in \mathbb{C}^{n \times n}} r(A - BXX^*B^*) = r[A, B], \quad (3.19)$$

$$\min_{X \in \mathbb{C}^{n \times n}} r(A - BXX^*B^*) = i_-(A) + r[A, B] - i_-(M), \quad (3.20)$$

$$\max_{X \in \mathbb{C}^{n \times n}} i_+(A - BXX^*B^*) = i_+(A), \quad (3.21)$$

$$\min_{X \in \mathbb{C}^{n \times n}} i_+(A - BXX^*B^*) = r[A, B] - i_-(M), \quad (3.22)$$

$$\max_{X \in \mathbb{C}^{n \times n}} i_-(A - BXX^*B^*) = i_-(M), \quad (3.23)$$

$$\min_{X \in \mathbb{C}^{n \times n}} i_-(A - BXX^*B^*) = i_-(A). \quad (3.24)$$

Lemma 3.2 ([9]) *Let $A_i \in \mathbb{C}_H^{m_i}$ and $B_i \in \mathbb{C}^{m_i \times n}$ be given, $i = 1, 2$, and assume that the matrix equation $B_2XB_2^* = A_2$ has a Hermitian solution. Also let*

$$\mathcal{S} = \{X \in \mathbb{C}_H^n \mid B_2XB_2^* = A_2\}, \quad M = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ B_1^* & B_2^* & 0 \end{bmatrix}, \quad N = \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \end{bmatrix}. \quad (3.25)$$

Then,

$$\max_{X \in \mathcal{S}} r(A_1 - B_1XB_1^*) = \min\{r[A_1, B_1], \quad r(M) - 2r(B_2)\}, \quad (3.26)$$

$$\min_{X \in \mathcal{S}} r(A_1 - B_1XB_1^*) = 2r[A_1, B_1] - 2r(N) + r(M), \quad (3.27)$$

$$\max_{X \in \mathcal{S}} i_{\pm}(A_1 - B_1XB_1^*) = i_{\pm}(M) - r(B_2), \quad (3.28)$$

$$\min_{X \in \mathcal{S}} i_{\pm}(A_1 - B_1XB_1^*) = r[A_1, B_1] - r(N) + i_{\pm}(M). \quad (3.29)$$

In consequence, the following hold.

(a) *There exists an $X \in \mathbb{C}_H^n$ such that $A_1 - B_1XB_1^*$ is nonsingular and $B_2XB_2^* = A_2$ if and only if $r[A_1, B_1] = m_1$ or $r(M) = 2r(B_2) + m_1$.*

(b) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* = A_1$ and $B_2XB_2^* = A_2$ if and only if*

$$\mathcal{R}(A_1) \subseteq \mathcal{R}(B_1), \quad \mathcal{R}(A_2) \subseteq \mathcal{R}(B_2), \quad r(M) = 2r \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \quad (3.30)$$

(c) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* \prec A_1$ and $B_2XB_2^* = A_2$ if and only if $i_+(M) = r(B_2) + m_1$.*

(d) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* \succ A_1$ and $B_2XB_2^* = A_2$ if and only if $i_-(M) = r(B_2) + m_1$.*

(e) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* \preceq A_1$ and $B_2XB_2^* = A_2$ if and only if*

$$\mathcal{R}(A_2) \subseteq \mathcal{R}(B_2) \quad \text{and} \quad i_+(M) = r(N) - r[A_1, B_1]. \quad (3.31)$$

(f) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* \succcurlyeq A_1$ and $B_2XB_2^* = A_2$ if and only if*

$$\mathcal{R}(A_2) \subseteq \mathcal{R}(B_2) \quad \text{and} \quad i_-(M) = r(N) - r[A_1, B_1]. \quad (3.32)$$

We next solve (1.8)–(1.11) for $i = 1$.

Theorem 3.3 *Let $A_i \in \mathbb{C}_H^{m_i}$ and $B_i \in \mathbb{C}^{m_i \times n}$ be given, $i = 1, 2$, and let M and N be of the forms in (3.25).*

Also, assume that there exists an $X \in \mathbb{C}_H^n$ such that $B_2XB_2^ \succcurlyeq A_2$, i.e., $i_- \begin{bmatrix} A_2 & B_2 \\ B_2^* & 0 \end{bmatrix} = r[A_2, B_2]$, and let*

$$\mathcal{T}_1 = \{X \in \mathbb{C}_H^n \mid B_2XB_2^* \succcurlyeq A_2\}, \quad M_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^* & 0 \end{bmatrix}. \quad (3.33)$$

Then,

$$\max_{X \in \mathcal{T}_1} r(A_1 - B_1XB_1^*) = r[A_1, B_1], \quad (3.34)$$

$$\min_{X \in \mathcal{T}_1} r(A_1 - B_1XB_1^*) = 2r[A_1, B_1] + i_-(M) - i_-(M_1) - r(N), \quad (3.35)$$

$$\max_{X \in \mathcal{T}_1} i_+(A_1 - B_1XB_1^*) = i_+(M) - r[A_2, B_2], \quad (3.36)$$

$$\min_{X \in \mathcal{T}_1} i_+(A_1 - B_1XB_1^*) = r[A_1, B_1] - i_-(M_1), \quad (3.37)$$

$$\max_{X \in \mathcal{T}_1} i_-(A_1 - B_1XB_1^*) = i_-(M_1), \quad (3.38)$$

$$\min_{X \in \mathcal{T}_1} i_-(A_1 - B_1XB_1^*) = r[A_1, B_1] - r(N) + i_-(M). \quad (3.39)$$

In consequence, the following hold.

(a) *There exists an $X \in \mathbb{C}_H^n$ such that $A_1 - B_1XB_1^*$ is nonsingular and $B_2XB_2^* \succcurlyeq A_2$ if and only if $r[A_1, B_1] = m_1$.*

(b) *The rank of $A_1 - B_1XB_1^*$ is nonsingular for any $B_2XB_2^* \succcurlyeq A_2$ if and only if $i_-(M) = i_-(M_1) + r(N) - m_1$.*

(c) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* = A_1$ and $B_2XB_2^* \succcurlyeq A_2$ if and only if $\mathcal{R}(A_1) \subseteq \mathcal{R}(B_1)$ and $i_-(M) = r \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$.*

(d) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* \prec A_1$ and $B_2XB_2^* \succcurlyeq A_2$ if and only if $i_+(M) = r[A_2, B_2] + m_1$.*

(e) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* \succ A_1$ and $B_2XB_2^* \succcurlyeq A_2$ if and only if $i_-(M_1) = m_1$.*

(f) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* \preceq A_1$ and $B_2XB_2^* \succcurlyeq A_2$ if and only if $i_-(M) = r(N) - r[A_1, B_1]$.*

(g) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1XB_1^* \succcurlyeq A_1$ and $B_2XB_2^* \succcurlyeq A_2$ if and only if $i_-(M_1) = r[A_1, B_1]$.*

(h) *$B_1XB_1^* \succcurlyeq A_1$ holds for all $X \in \mathbb{C}_H^n$ such that $B_2XB_2^* \succcurlyeq A_2$, i.e.,*

$$\{X \in \mathbb{C}_H^n \mid B_2XB_2^* \succcurlyeq A_2\} \subseteq \{X \in \mathbb{C}_H^n \mid B_1XB_1^* \succcurlyeq A_1\}$$

if and only if $i_+(M) = r[A_2, B_2]$.

(i) *The rank of $A_1 - B_1XB_1^*$ is invariant with respect to the Hermitian solution of $B_2XB_2^* \succcurlyeq A_2$ if and only if $i_-(M) = i_-(M_1) + r(N) - r[A_1, B_1]$.*

- (j) The positive index of the inertia of $A_1 - B_1 X B_1^*$ is invariant with respect to the solution of $B_2 X B_2^* \succcurlyeq A_2$ if and only if $i_+(M) + i_-(M_1) = r[A_1, B_1] + r[A_2, B_2]$.
- (k) The negative index of the inertia of $A_1 - B_1 X B_1^*$ is invariant with respect to the solution of $B_2 X B_2^* \succcurlyeq A_2$ if and only if $i_-(M_1) + r(N) = r[A_1, B_1] + i_-(M)$.
- (l) If there exist $X_1, X_2 \in \mathbb{C}_H^n$ such that $B_1 X_1 B_1^* = A_1$ and $B_2 X_2 B_2^* = A_2$ hold, respectively, i.e., $\mathcal{R}(A_1) \subseteq \mathcal{R}(B_1)$ and $\mathcal{R}(A_2) \subseteq \mathcal{R}(B_2)$, then,

$$\max_{X \in \mathcal{T}_1} r(A_1 - B_1 X B_1^*) = r(B_1), \quad (3.40)$$

$$\min_{X \in \mathcal{T}_1} r(A_1 - B_1 X B_1^*) = i_-(M) - r[B_1, B_2], \quad (3.41)$$

$$\max_{X \in \mathcal{T}_1} i_+(A_1 - B_1 X B_1^*) = i_+(M) - r(B_2), \quad (3.42)$$

$$\min_{X \in \mathcal{T}_1} i_+(A_1 - B_1 X B_1^*) = 0, \quad (3.43)$$

$$\max_{X \in \mathcal{T}_1} i_-(A_1 - B_1 X B_1^*) = r(B_1), \quad (3.44)$$

$$\min_{X \in \mathcal{T}_1} i_-(A_1 - B_1 X B_1^*) = i_-(M) - r[B_1, B_2]. \quad (3.45)$$

Proof. From Theorem 2.2(a), the general Hermitian solution of $B_2 X B_2^* \succcurlyeq A_2$ can be written as

$$X = [0, I_n] J^\dagger \begin{bmatrix} 0 \\ I_n \end{bmatrix} + U U^* - F_{B_2} V - V^* F_{B_2}, \quad J = \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}, \quad (3.46)$$

where $U, V \in \mathbb{C}^{n \times n}$ are arbitrary. Substituting (3.46) into $A_1 - B_1 X B_1^*$ gives

$$A_1 - B_1 X B_1^* = \hat{A} - B_1 U U^* B_1^* + B_1 F_{B_2} V B_1^* + B_1 V^* F_{B_2} B_1^*, \quad (3.47)$$

where $\hat{A} = A_1 - [0, B_1] J^\dagger \begin{bmatrix} 0 \\ B_1^* \end{bmatrix}$. In consequence,

$$\max_{X \in \mathcal{T}_1} r(A_1 - B_1 X B_1^*) = \max_{U, V \in \mathbb{C}^{n \times n}} r(\hat{A} - B_1 U U^* B_1^* + B_1 F_{B_2} V B_1^* + B_1 V^* F_{B_2} B_1^*), \quad (3.48)$$

$$\min_{X \in \mathcal{T}_1} r(A_1 - B_1 X B_1^*) = \min_{U, V \in \mathbb{C}^{n \times n}} r(\hat{A} - B_1 U U^* B_1^* + B_1 F_{B_2} V B_1^* + B_1 V^* F_{B_2} B_1^*), \quad (3.49)$$

$$\max_{X \in \mathcal{T}_1} i_\pm(A_1 - B_1 X B_1^*) = \max_{U, V \in \mathbb{C}^{n \times n}} i_\pm(\hat{A} - B_1 U U^* B_1^* + B_1 F_{B_2} V B_1^* + B_1 V^* F_{B_2} B_1^*), \quad (3.50)$$

$$\min_{X \in \mathcal{T}_1} i_\pm(A_1 - B_1 X B_1^*) = \min_{U, V \in \mathbb{C}^{n \times n}} i_\pm(\hat{A} - B_1 U U^* B_1^* + B_1 F_{B_2} V B_1^* + B_1 V^* F_{B_2} B_1^*). \quad (3.51)$$

Applying (1.39)–(1.42) gives

$$\begin{aligned} & \max_{V \in \mathbb{C}^{n \times n}} r(\hat{A} - B_1 U U^* B_1^* + B_1 F_{B_2} V B_1^* + B_1 V^* F_{B_2} B_1^*) \\ &= \min \left\{ r[A_1, B_1], \quad r \begin{bmatrix} \hat{A} - B_1 U U^* B_1^* & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} \right\}, \end{aligned} \quad (3.52)$$

$$\begin{aligned} & \min_{V \in \mathbb{C}^{n \times n}} r(\hat{A} - B_1 U U^* B_1^* + B_1 F_{B_2} V B_1^* + B_1 V^* F_{B_2} B_1^*) \\ &= 2r[A_1, B_1] + r \begin{bmatrix} \hat{A} - B_1 U U^* B_1^* & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} - 2r \begin{bmatrix} A_1 & B_1 F_{B_2} \\ B_1^* & 0 \end{bmatrix}, \end{aligned} \quad (3.53)$$

$$\max_{V \in \mathbb{C}^{n \times n}} i_\pm(\hat{A} - B_1 U U^* B_1^* + B_1 F_{B_2} V B_1^* + B_1 V^* F_{B_2} B_1^*) = i_\pm \begin{bmatrix} \hat{A} - B_1 U U^* B_1^* & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix}, \quad (3.54)$$

$$\begin{aligned} & \min_{V \in \mathbb{C}^{n \times n}} i_\pm(\hat{A} - B_1 U U^* B_1^* + B_1 F_{B_2} V B_1^* + B_1 V^* F_{B_2} B_1^*) \\ &= r[A_1, B_1] + i_\pm \begin{bmatrix} \hat{A} - B_1 U U^* B_1^* & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} - r \begin{bmatrix} A_1 & B_1 F_{B_2} \\ B_1^* & 0 \end{bmatrix}. \end{aligned} \quad (3.55)$$

Note that

$$\begin{bmatrix} \hat{A} - B_1 U U^* B_1^* & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} = \begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} U U^* [B_1^*, 0]. \quad (3.56)$$

Then applying (3.7)–(3.12) and simplifying by (1.19) and (1.28), we obtain

$$\begin{aligned} & \max_{U \in \mathbb{C}^{n \times n}} r \left(\begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} U U^* [B_1^*, 0] \right) \\ &= r \begin{bmatrix} \hat{A} & B_1 F_{B_2} & B_1 \\ F_{B_2} B_1^* & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_1 & B_1 \\ F_{B_2} B_1^* & 0 \end{bmatrix} = r \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \end{bmatrix} - r(B_2) = r(N) - r(B_2), \end{aligned} \quad (3.57)$$

$$\begin{aligned} & \min_{U \in \mathbb{C}^{n \times n}} r \left(\begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} U U^* [B_1^*, 0] \right) \\ &= i_- \begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} + r \begin{bmatrix} A_1 & B_1 F_{B_2} & B_1 \\ F_{B_2} B_1^* & 0 & 0 \end{bmatrix} - i_- \begin{bmatrix} A_1 & B_1 F_{B_2} & B_1 \\ F_{B_2} B_1^* & 0 & 0 \end{bmatrix} \\ &= i_- \begin{bmatrix} \hat{A} & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & 0 \end{bmatrix} + r \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \end{bmatrix} - 2r(B_2) - i_- \begin{bmatrix} A_1 & B_1 \\ B_1^* & 0 \end{bmatrix}, \end{aligned} \quad (3.58)$$

$$\begin{aligned} & \max_{U \in \mathbb{C}^{n \times n}} i_+ \left(\begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} U U^* [B_1^*, 0] \right) = i_+ \begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} \\ &= i_+ \begin{bmatrix} \hat{A} & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & 0 \end{bmatrix} - r(B_2), \end{aligned} \quad (3.59)$$

$$\begin{aligned} & \min_{U \in \mathbb{C}^{n \times n}} i_+ \left(\begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} U U^* [B_1^*, 0] \right) \\ &= r \begin{bmatrix} \hat{A} & B_1 F_{B_2} & B_1 \\ F_{B_2} B_1^* & 0 & 0 \end{bmatrix} - i_- \begin{bmatrix} \hat{A} & B_1 F_{B_2} & B_1 \\ F_{B_2} B_1^* & 0 & 0 \end{bmatrix} \\ &= r \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \end{bmatrix} - r(B_2) - i_- \begin{bmatrix} A_1 & B_1 \\ B_1^* & 0 \end{bmatrix}, \end{aligned} \quad (3.60)$$

$$\begin{aligned} & \max_{U \in \mathbb{C}^{n \times n}} i_- \left(\begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} U U^* [B_1^*, 0] \right) = i_- \begin{bmatrix} \hat{A} & B_1 F_{B_2} & B_1 \\ F_{B_2} B_1^* & 0 & 0 \\ B_1^* & 0 & 0 \end{bmatrix} \\ &= i_- \begin{bmatrix} A_1 & B_1 \\ B_1^* & 0 \end{bmatrix}, \end{aligned} \quad (3.61)$$

$$\begin{aligned} & \min_{U \in \mathbb{C}^{n \times n}} i_- \left(\begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} U U^* [B_1^*, 0] \right) = i_- \begin{bmatrix} \hat{A} & B_1 F_{B_2} \\ F_{B_2} B_1^* & 0 \end{bmatrix} \\ &= i_- \begin{bmatrix} \hat{A} & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & 0 \end{bmatrix} - r(B_2). \end{aligned} \quad (3.62)$$

Further, applying congruence matrix operations gives

$$\begin{aligned} & i_{\pm} \begin{bmatrix} \hat{A} & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & 0 \end{bmatrix} \\ &= i_{\pm} \begin{bmatrix} A_1 - [0, B_1] J^{\dagger} \begin{bmatrix} 0 \\ B_1^* \end{bmatrix} & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & 0 \end{bmatrix} = i_{\pm} \begin{bmatrix} A_1 & B_1 & \frac{1}{2}[0, B_1] J^{\dagger} \begin{bmatrix} 0 \\ B_2^* \end{bmatrix} \\ B_1^* & 0 & B_2^* \\ \frac{1}{2}[0, B_2] J^{\dagger} \begin{bmatrix} 0 \\ B_1^* \end{bmatrix} & B_2 & 0 \end{bmatrix} \\ &= i_{\pm} \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & -[0, B_2] J^{\dagger} \begin{bmatrix} 0 \\ B_2^* \end{bmatrix} \end{bmatrix} = i_{\pm} \left(\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ [0, B_2] \end{bmatrix} J^{\dagger} \begin{bmatrix} 0, 0, \begin{bmatrix} 0 \\ B_2^* \end{bmatrix} \end{bmatrix} \right) \end{aligned} \quad (3.63)$$

$$\begin{aligned}
&= i_{\pm} \begin{bmatrix} A_1 & B_1 & 0 & 0 & 0 \\ B_1^* & 0 & B_2^* & 0 & 0 \\ 0 & B_2 & 0 & 0 & B_2 \\ 0 & 0 & 0 & -A_2 & B_2 \\ 0 & 0 & B_2^* & B_2^* & 0 \end{bmatrix} - i_{\pm}(J) \quad (\text{by (1.27)}) = i_{\pm} \begin{bmatrix} A_1 & B_1 & 0 & 0 & 0 \\ B_1^* & 0 & 0 & -B_2^* & 0 \\ 0 & 0 & 0 & 0 & B_2 \\ 0 & -B_2 & 0 & -A_2 & 0 \\ 0 & 0 & B_2^* & 0 & 0 \end{bmatrix} - i_{\pm}(J) \\
&= i_{\pm} \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^* & 0 & B_2^* \\ 0 & B_2 & -A_2 \end{bmatrix} + r(B_2) - i_{\pm}(J) \quad (\text{by (1.17) and (1.18)}) \\
&= i_{\pm}(M) + r(B_2) - i_{\mp} \begin{bmatrix} A_2 & B_2 \\ B_2^* & 0 \end{bmatrix} \quad (\text{by (1.15) and (1.16)}). \tag{3.64}
\end{aligned}$$

Substituting (3.64) into (3.57)–(3.62), and then (3.57)–(3.62) into (3.52)–(3.55) yields (3.34)–(3.39). Applying Lemma 1.4 to (3.34)–(3.39) leads to (a)–(g). Result (h) follows from (3.34)–(3.39). \square

Similarly, we can show the following result.

Theorem 3.4 *Let $A_i \in \mathbb{C}_H^{m_i}$ and $B_i \in \mathbb{C}^{m_i \times n}$ be given, $i = 1, 2$, and let M and N be of the forms in (3.25). Also, assume that there exists an $X \in \mathbb{C}_H^n$ such that $B_2 X B_2^* \preccurlyeq A_2$, i.e., $i_+ \begin{bmatrix} A_2 & B_2 \\ B_2^* & 0 \end{bmatrix} = r[A_2, B_2]$, and let*

$$M_1 = \begin{bmatrix} A_1 & B_1 \\ B_1^* & 0 \end{bmatrix}, \quad \mathcal{T}_2 = \{X \in \mathbb{C}_H^n \mid B_2 X B_2^* \preccurlyeq A_2\}. \tag{3.65}$$

Then,

$$\max_{X \in \mathcal{T}_2} r(A_1 - B_1 X B_1^*) = r[A_1, B_1], \tag{3.66}$$

$$\min_{X \in \mathcal{T}_2} r(A_1 - B_1 X B_1^*) = 2r[A_1, B_1] + i_+(M) - i_+(M_1) - r(N), \tag{3.67}$$

$$\max_{X \in \mathcal{T}_2} i_+(A_1 - B_1 X B_1^*) = i_+(M_1), \tag{3.68}$$

$$\min_{X \in \mathcal{T}_2} i_+(A_1 - B_1 X B_1^*) = r[A_1, B_1] - r(N) + i_+(M), \tag{3.69}$$

$$\max_{X \in \mathcal{T}_2} i_-(A_1 - B_1 X B_1^*) = i_-(M) - r[A_2, B_2], \tag{3.70}$$

$$\min_{X \in \mathcal{T}_2} i_-(A_1 - B_1 X B_1^*) = r[A_1, B_1] - i_+(M_1). \tag{3.71}$$

In consequence, the following hold.

- (a) *There exists an $X \in \mathbb{C}_H^n$ such that $A_1 - B_1 X B_1^*$ is nonsingular and $B_2 X B_2^* \preccurlyeq A_2$ if and only if $r[A_1, B_1] = m_1$.*
- (b) *The rank of $A_1 - B_1 X B_1^*$ is nonsingular for any $B_2 X B_2^* \preccurlyeq A_2$ if and only if $i_+(M) = i_+(M_1) + r(N) - m_1$.*
- (c) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1 X B_1^* = A_1$ and $B_2 X B_2^* \preccurlyeq A_2$ if and only if $\mathcal{R}(A_1) \subseteq \mathcal{R}(B_1)$ and $i_+(M) = r \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$.*
- (d) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1 X B_1^* \prec A_1$ and $B_2 X B_2^* \preccurlyeq A_2$ if and only if $i_+(M_1) = m_1$.*
- (e) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1 X B_1^* \succ A_1$ and $B_2 X B_2^* \preccurlyeq A_2$ if and only if $i_-(M) = r[A_2, B_2] + m_1$.*
- (f) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1 X B_1^* \preccurlyeq A_1$ and $B_2 X B_2^* \preccurlyeq A_2$ if and only if $i_+(M_1) = r[A_1, B_1]$.*
- (g) *There exists an $X \in \mathbb{C}_H^n$ such that $B_1 X B_1^* \succcurlyeq A_1$ and $B_2 X B_2^* \preccurlyeq A_2$ if and only if $i_+(M) = r(N) - r[A_1, B_1]$.*
- (h) *$B_1 X B_1^* \preccurlyeq A_1$ holds for all $X \in \mathbb{C}_H^n$ such that $B_2 X B_2^* \preccurlyeq A_2$, i.e.,*

$$\{X \in \mathbb{C}_H^n \mid B_2 X B_2^* \preccurlyeq A_2\} \subseteq \{X \in \mathbb{C}_H^n \mid B_1 X B_1^* \preccurlyeq A_1\}$$

if and only if $i_-(M) = r[A_2, B_2]$.

- (i) *The rank of $A_1 - B_1 X B_1^*$ is invariant with respect to the Hermitian solution of $B_2 X B_2^* \preccurlyeq A_2$ if and only if $i_+(M) = i_+(M_1) + r(N) - r[A_1, B_1]$.*
- (j) *The positive index of the inertia of $A_1 - B_1 X B_1^*$ is invariant with respect to the Hermitian solution of $B_2 X B_2^* \preccurlyeq A_2$ if and only if $i_+(M_1) + r(N) = r[A_1, B_1] + i_+(M)$.*
- (k) *The negative index of the inertia of $A_1 - B_1 X B_1^*$ is invariant with respect to the Hermitian solution of $B_2 X B_2^* \preccurlyeq A_2$ if and only if $i_-(M) + i_+(M_1) = r[A_1, B_1] + r[A_2, B_2]$.*

Theorem 3.3(g) shows that the pair of LMIs $B_1XB_1^* \succcurlyeq A_1$ and $B_2XB_2^* \succcurlyeq A_2$ have a common Hermitian solution if and only if

$$i_- \begin{bmatrix} A_1 & B_1 \\ B_1^* & 0 \end{bmatrix} = r[A_1, B_1], \quad i_- \begin{bmatrix} A_2 & B_2 \\ B_2^* & 0 \end{bmatrix} = r[A_2, B_2],$$

namely, $B_1XB_1^* \succcurlyeq A_1$ and $B_2XB_2^* \succcurlyeq A_2$ have a Hermitian solution, respectively. This simple fact makes us to give the following conjecture.

Conjecture 3.5 *The k LMIs*

$$B_1XB_1^* \succcurlyeq A_1, \dots, B_kXB_k^* \succcurlyeq A_k$$

have a common Hermitian solution if and only if each of the k LMIs has a Hermitian solution.

4 Global maximal and minimal matrices of $A_1 - B_1XB_1^*$ subject to LMIs

In this section, we solve the two LMI-constrained partial ordering optimization problems in (1.13). Let

$$\phi(X) = A_1 - B_1XB_1^*. \quad (4.1)$$

Then, (1.13) is equivalent to finding $X_1, X_2 \in \mathbb{C}_H^n$ such that

$$B_2X_1B_2^* \succcurlyeq A_2 \quad \text{and} \quad \phi(X_1) \succcurlyeq \phi(X) \quad \text{for all solutions of} \quad B_2XB_2^* \succcurlyeq A_2, \quad (4.2)$$

$$B_2X_2B_2^* \preccurlyeq A_2 \quad \text{and} \quad \phi(X_2) \preccurlyeq \phi(X) \quad \text{for all solutions of} \quad B_2XB_2^* \preccurlyeq A_2 \quad (4.3)$$

hold, respectively.

Theorem 4.1 *Let $A_i \in \mathbb{C}_H^{m_i}$ and $B_i \in \mathbb{C}^{m_i \times n}$ be given for $i = 1, 2$, and assume that $B_2XB_2^* \succcurlyeq A_2$ is consistent. Then, (1.13) has a solution if and only if*

$$\mathcal{R} \begin{bmatrix} 0 \\ B_1^* \end{bmatrix} \subseteq \mathcal{R} \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}. \quad (4.4)$$

In this case, the global maximizer $X_0 \in \mathbb{C}_H^n$ of (4.2) is given by

$$X_0 = [0, I_n] \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \quad (4.5)$$

and the global maximal matrix in (1.13) can be written as

$$\max_{\succcurlyeq} \{ A_1 - B_1XB_1^* \mid B_2XB_2^* \succcurlyeq A_2, X \in \mathbb{C}_H^n \} = A_1 - [0, B_1] \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ B_1^* \end{bmatrix}, \quad (4.6)$$

which satisfies

$$i_\pm \left(A_1 - [0, B_1] \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ B_1^* \end{bmatrix} \right) = i_\pm(M) - i_\pm \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}. \quad (4.7)$$

Proof. From Lemma 1.4(i), there exists a Hermitian solution X_0 of $B_2XB_2^* \succcurlyeq A_2$ such that (4.2) holds if and only if

$$\max_{B_2XB_2^* \succcurlyeq A_2} i_+[\phi(X) - \phi(X_0)] = 0. \quad (4.8)$$

Note that $\phi(X) - \phi(X_0) = B_1X_0B_1^* - B_1XB_1^*$. Applying (3.36) to it gives

$$\begin{aligned} \max_{B_2XB_2^* \succcurlyeq A_2} i_+[\phi(X) - \phi(X_0)] &= \max_{B_2XB_2^* \succcurlyeq A_2} i_+(B_1X_0B_1^* - B_1XB_1^*) \\ &= i_+ \begin{bmatrix} B_1X_0B_1^* & 0 & B_1 \\ 0 & -A_2 & B_2 \\ B_1^* & B_2^* & 0 \end{bmatrix} - i_+ \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}. \end{aligned} \quad (4.9)$$

Substituting (4.9) into (4.8) leads to

$$i_+ \begin{bmatrix} B_1X_0B_1^* & 0 & B_1 \\ 0 & -A_2 & B_2 \\ B_1^* & B_2^* & 0 \end{bmatrix} = i_+ \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix},$$

which, by Lemma 1.7(g), is equivalent to

$$\mathcal{R} \begin{bmatrix} 0 \\ B_1^* \end{bmatrix} \subseteq \mathcal{R} \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix} \quad \text{and} \quad B_1 X_0 B_1^* \preccurlyeq [0, B_1] \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ B_1^* \end{bmatrix}. \quad (4.10)$$

The general solution of the matrix in (4.10) can be derived from Corollary 2.3(a). Comparing the inequality with (3.46), we obtain the special solution in (4.5). \square

The following result can be shown similarly.

Theorem 4.2 *Let $A_i \in \mathbb{C}_H^{m_i}$ and $B_i \in \mathbb{C}^{m_i \times n}$ be given for $i = 1, 2$, and assume that $B_2 X B_2^* \preccurlyeq A_2$ is consistent. Then, the second problem in (1.13) has a solution if and only if*

$$\mathcal{R} \begin{bmatrix} 0 \\ B_1^* \end{bmatrix} \subseteq \mathcal{R} \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}. \quad (4.11)$$

In this case, the global minimizer $X_0 \in \mathbb{C}_H^n$ of (4.2) is given by

$$X_0 = [0, I_n] \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \quad (4.12)$$

and the global minimal matrix in the second problem of (1.13) can uniquely be written as

$$\min_{\succcurlyeq} \{ A_1 - B_1 X B_1^* \mid B_2 X B_2^* \preccurlyeq A_2 \} = A_1 - [0, B_1] \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ B_1^* \end{bmatrix}, \quad (4.13)$$

which satisfies

$$i_\pm \left(A_1 - [0, B_1] \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ B_1^* \end{bmatrix} \right) = i_\pm(M) - i_\pm \begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}. \quad (4.14)$$

It is obvious that the right-hand sides of (4.9) and (4.13) are the Schur complement of $\begin{bmatrix} -A_2 & B_2 \\ B_2^* & 0 \end{bmatrix}$ in the adjoint block matrix M in (3.25).

5 Ranks and inertias of the Hermitian solutions of $BXB^* \succcurlyeq A$

Note from (2.8) and (2.24) that the Hermitian solutions of (2.5) and (2.21) are in fact quadratic matrix-valued functions that involve two variable matrices. Hence, we are able to derive from Lemma 1.11 and Theorem 2.6 a group of formulas for calculating the global maximal and minimal ranks and inertias of the Hermitian solutions of the two LMIs in (2.5) and (2.21).

Theorem 5.1 *Let $A \in \mathbb{C}_H^m$ and $B \in \mathbb{C}^{m \times n}$ be given.*

(a) *If there exists an $X \in \mathbb{C}_H^n$ such that $BXB^* \succcurlyeq A$ holds, then,*

$$\max_{BXB^* \succcurlyeq A} r(X) = \max_{BXB^* \succcurlyeq A} i_+(X) = n, \quad (5.1)$$

$$\min_{BXB^* \succcurlyeq A} r(X) = \min_{BXB^* \succcurlyeq A} i_+(X) = i_+(A), \quad (5.2)$$

$$\max_{BXB^* \succcurlyeq A} i_-(X) = n + i_-(A) - r[A, B], \quad (5.3)$$

$$\min_{BXB^* \succcurlyeq A} i_-(X) = 0. \quad (5.4)$$

In consequence, the following hold.

- (i) The matrix \hat{X} in (2.31) is a solution that satisfies the second equality in (5.1).
- (ii) $BXB^* \succcurlyeq A$ always has a solution $X \succ 0$.
- (iii) All solutions of $BXB^* \succcurlyeq A$ satisfy $X \succ 0$ if and only if $i_+(A) = n$.
- (iv) $X = 0$ is a solution of $BXB^* \succcurlyeq A \Leftrightarrow BXB^* \succcurlyeq A$ has a solution $X \preccurlyeq 0 \Leftrightarrow A \preccurlyeq 0$.
- (v) $BXB^* \succcurlyeq A$ has a solution $X \prec 0$ if and only if $A \prec 0$.
- (vi) All solutions of $BXB^* \succcurlyeq A$ satisfy $X \succcurlyeq 0$ if and only if $r(B) = n$ and $r[A, B] = i_-(A) + n$.

(b) If there exists an $X \in \mathbb{C}_H^n$ such that $BXB^* \preceq A$, then,

$$\max_{BXB^* \preceq A} r(X) = \max_{BXB^* \preceq A} i_-(X) = n, \quad (5.5)$$

$$\min_{BXB^* \preceq A} r(X) = \min_{BXB^* \preceq A} i_-(X) = i_-(A), \quad (5.6)$$

$$\max_{BXB^* \preceq A} i_+(X) = n + i_+(A) - r[A, B], \quad (5.7)$$

$$\min_{BXB^* \preceq A} i_+(X) = 0. \quad (5.8)$$

In consequence, the following hold.

- (i) The matrix \hat{X} in (2.31) is a solution that satisfies the second equality in (5.5).
- (ii) $BXB^* \preceq A$ always has a solution $X \prec 0$.
- (iii) All solutions of $BXB^* \preceq A$ satisfy $X \prec 0$ if and only if $i_-(A) = n$.
- (iv) $X = 0$ is a solution of $BXB^* \preceq A \Leftrightarrow BXB^* \preceq A$ has a solution $X \succ 0 \Leftrightarrow A \succ 0$.
- (v) $BXB^* \preceq A$ has a solution $X \succ 0$ if and only if $A \succ 0$.
- (vi) All solutions of $BXB^* \preceq A$ satisfy $X \preceq 0$ if and only if $r(B) = n$ and $r[A, B] = i_+(A) + n$.

Theorem 5.1 shows the ranks and inertias of Hermitian solutions of the two simple LMIs $BXB^* \succcurlyeq A$ and $BXB^* \preceq A$ may have different values. In general, solutions of LMEs and LMIs with low ranks or inertias are objects of particular interest in the investigations of LMEs and LMIs and their applications; some recent work on this topic can be found, e.g., in [1, 13].

We now turn our attention to the ranks and inertias of submatrices in Hermitian solutions of the matrix equation $BXB^* = A$ and the inequality $BXB^* \succcurlyeq A$ ($BXB^* \preceq A$). Rewrite $BXB^* \succcurlyeq A$ as

$$[B_1, B_2] \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} = A, \quad (5.9)$$

where $B_1 \in \mathbb{C}^{m \times n_1}$, $B_2 \in \mathbb{C}^{m \times n_2}$, $X_1 \in \mathbb{C}_H^{n_1}$, $X_2 \in \mathbb{C}^{n_1 \times n_2}$ and $X_3 \in \mathbb{C}_H^{n_2}$ with $n_1 + n_2 = n$. Note that the submatrices X_1, X_2, X_3 in (5.9) can be rewritten as

$$X_1 = P_1 X P_1^*, \quad X_2 = P_1 X P_2^*, \quad X_3 = P_2 X P_2^*, \quad (5.10)$$

where $P_1 = [I_{n_1}, 0]$ and $P_2 = [0, I_{n_2}]$. For convenience, we adopt the following notation for the collections of the submatrices X_1 and X_3 in (5.9):

$$\mathcal{S}_1 = \left\{ X_1 \in \mathbb{C}_H^{n_1} \mid [B_1, B_2] \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} = A \right\} = \{X_1 = P_1 X P_1^* \mid BXB^* = A, X \in \mathbb{C}_H^n\}, \quad (5.11)$$

$$\mathcal{S}_3 = \left\{ X_3 \in \mathbb{C}_H^{n_2} \mid [B_1, B_2] \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} = A \right\} = \{X_3 = P_2 X P_2^* \mid BXB^* = A, X \in \mathbb{C}_H^n\}. \quad (5.12)$$

Applying Lemma 3.2 to (5.11) and (5.12) and simplifying, we obtain the global maximal and minimal ranks and inertias of the submatrices X_1 and X_3 in a Hermitian solution of (5.9) and their consequences as follows. The details of the proof are omitted.

Theorem 5.2 Assume that the matrix equation (5.9) is consistent, and let \mathcal{S}_1 and \mathcal{S}_3 be of the forms in (5.11) and (5.12). Then,

$$\max_{X_1 \in \mathcal{S}_1} r(X_1) = \min \left\{ n_1, r \begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - 2r(B) + 2n_1 \right\}, \quad (5.13)$$

$$\min_{X_1 \in \mathcal{S}_1} r(X_1) = r \begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - 2r(B_2), \quad (5.14)$$

$$\max_{X_1 \in \mathcal{S}_1} i_{\pm}(X_1) = i_{\pm} \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} - r(A) + n_1, \quad (5.15)$$

$$\min_{X_1 \in \mathcal{S}_1} i_{\pm}(X_1) = i_{\pm} \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} - r(B_2), \quad (5.16)$$

$$\max_{X_3 \in \mathcal{S}_3} r(X_3) = \min \left\{ n_2, r \begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - 2r(B) + 2n_2 \right\}, \quad (5.17)$$

$$\min_{X_3 \in \mathcal{S}_3} r(X_3) = r \begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - 2r(B_1), \quad (5.18)$$

$$\max_{X_3 \in \mathcal{S}_3} i_{\pm}(X_3) = i_{\pm} \begin{bmatrix} C & B_1 \\ A_1^* & 0 \end{bmatrix} - r(A) + n_2, \quad (5.19)$$

$$\min_{X_3 \in \mathcal{S}_3} i_{\pm}(X_3) = i_{\pm} \begin{bmatrix} C & B_1 \\ A_1^* & 0 \end{bmatrix} - r(B_1). \quad (5.20)$$

In consequence, the following hold.

- (c) Eq. (5.9) has a Hermitian solution in which X_1 is nonsingular if and only if $r \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} \geq 2r(A) - n_1$.
- (d) The submatrix X_1 in any Hermitian solution of (5.9) is nonsingular if and only if $r \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = 2r(B_2) + n_1$.
- (e) Eq. (5.9) has a Hermitian solution in which $X_1 = 0$ if and only if $r \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = 2r(B_2)$.
- (f) The submatrix X_1 in any Hermitian solution of (5.9) satisfies $X_1 = 0$ if and only if $r \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = 2r(A) - 2n_1$.
- (g) Eq. (5.10) has a Hermitian solution in which $X_1 \succ 0$ ($X_1 \prec 0$) if and only if

$$i_+ \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(A) \quad \left(i_- \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(A) \right).$$

- (h) The submatrix X_1 in any Hermitian solution of (5.9) satisfies $X_1 \succ 0$ ($X_1 \prec 0$) if and only if

$$i_+ \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = n_1 + r(B_2) \quad \left(i_- \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = n_1 + r(B_2) \right).$$

- (i) Eq. (5.9) has a Hermitian solution satisfying $X_1 \succcurlyeq 0$ ($X_1 \preccurlyeq 0$) if and only if

$$i_- \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(B_2) \quad \left(i_+ \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(B_2) \right).$$

- (j) The submatrix X_1 in any Hermitian solution of (5.9) satisfies $X_1 \succcurlyeq 0$ ($X_1 \preccurlyeq 0$) if and only if

$$i_- \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(A) - n_1 \quad \left(i_+ \begin{bmatrix} C & B_2 \\ A_2^* & 0 \end{bmatrix} = r(A) - n_1 \right).$$

- (k) The positive signature of X_1 in (5.9) is invariant \Leftrightarrow the negative signature of X_1 (5.9) is invariant $\Leftrightarrow \mathcal{R}(B_1) \cap \mathcal{R}(B_2) = \{0\}$ and $r(B_1) = n_1$.

Replacing the equality sign in (5.9) with inequality signs gives

$$[B_1, B_2] \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} \succcurlyeq A, \quad (5.21)$$

$$[B_1, B_2] \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} \preccurlyeq A. \quad (5.22)$$

Also let

$$\mathcal{U}_1 = \left\{ X_1 \in \mathbb{C}_H^{n_1} \mid [B_1, B_2] \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} \succcurlyeq A \right\} = \{X_1 = P_1 X P_1^* \mid B X B^* \succcurlyeq A, X \in \mathbb{C}_H^n\}, \quad (5.23)$$

$$\mathcal{U}_3 = \left\{ X_3 \in \mathbb{C}_H^{n_2} \mid [B_1, B_2] \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} \succcurlyeq A \right\} = \{X_2 = P_2 X P_2^* \mid B X B^* \succcurlyeq A, X \in \mathbb{C}_H^n\}, \quad (5.24)$$

$$\mathcal{V}_1 = \left\{ X_1 \in \mathbb{C}_H^{n_1} \mid [B_1, B_2] \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} \preccurlyeq A \right\} = \{X_1 = P_1 X P_1^* \mid B X B^* \preccurlyeq A, X \in \mathbb{C}_H^n\}, \quad (5.25)$$

$$\mathcal{V}_3 = \left\{ X_3 \in \mathbb{C}_H^{n_2} \mid [B_1, B_2] \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} \preccurlyeq A \right\} = \{X_3 = P_2 X P_2^* \mid B X B^* \preccurlyeq A, X \in \mathbb{C}_H^n\}. \quad (5.26)$$

Applying Theorems 3.3 and 3.4 to (5.21)–(5.26) and simplifying, we obtain the global maximal and minimal ranks and inertias of the submatrices X_1 and X_3 in Hermitian solutions of (5.21) and (5.22) as follows. The details of the proof are omitted.

Theorem 5.3 Assume that the matrix inequality in (5.21) is consistent, and let \mathcal{U}_1 and \mathcal{U}_3 be of the forms in (5.23) and (5.24). Then,

$$\max_{X_1 \in \mathcal{U}_1} r(X_1) = \max_{X_1 \in \mathcal{U}_1} i_+(X_1) = n_1, \quad (5.27)$$

$$\min_{X_1 \in \mathcal{U}_1} r(X_1) = n_1 + i_+ \begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - r(B_1) - r(B_2), \quad (5.28)$$

$$\min_{X_1 \in \mathcal{U}_1} i_+(X_1) = i_+ \begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - r(B_2), \quad (5.29)$$

$$\max_{X_1 \in \mathcal{U}_1} i_-(X_1) = n_1 + i_- \begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - r[A, B], \quad (5.30)$$

$$\min_{X_1 \in \mathcal{U}_1} i_-(X_1) = 0, \quad (5.31)$$

$$\max_{X_3 \in \mathcal{U}_3} r(X_3) = \max_{X_3 \in \mathcal{U}_3} i_+(X_3) = n_2, \quad (5.32)$$

$$\min_{X_3 \in \mathcal{U}_3} r(X_3) = n_2 + i_+ \begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - r(B_1) - r(B_2), \quad (5.33)$$

$$\min_{X_3 \in \mathcal{U}_3} i_+(X_3) = i_+ \begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - r(B_1), \quad (5.34)$$

$$\max_{X_3 \in \mathcal{U}_3} i_-(X_3) = n_2 + i_- \begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - r[A, B], \quad (5.35)$$

$$\min_{X_3 \in \mathcal{U}_3} i_-(X_3) = 0. \quad (5.36)$$

Theorem 5.4 Assume that the matrix inequality in (5.22) is consistent, and let \mathcal{V}_1 and \mathcal{V}_3 be of the forms in (5.25) and (5.26). Then,

$$\max_{X_1 \in \mathcal{V}_1} r(X_1) = \max_{X_1 \in \mathcal{V}_1} i_-(X_1) = n_1, \quad (5.37)$$

$$\min_{X_1 \in \mathcal{V}_1} r(X_1) = n_1 + i_- \begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - r(B_1) - r(B_2), \quad (5.38)$$

$$\max_{X_1 \in \mathcal{V}_1} i_+(X_1) = n_1 + i_+ \begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - r[A, B], \quad (5.39)$$

$$\min_{X_1 \in \mathcal{V}_1} i_+(X_1) = 0, \quad (5.40)$$

$$\min_{X_1 \in \mathcal{V}_1} i_-(X_1) = i_- \begin{bmatrix} A & B_2 \\ B_2^* & 0 \end{bmatrix} - r(B_2), \quad (5.41)$$

$$\max_{X_3 \in \mathcal{V}_3} r(X_3) = \max_{X_3 \in \mathcal{V}_3} i_-(X_3) = n_2, \quad (5.42)$$

$$\min_{X_3 \in \mathcal{V}_3} r(X_3) = n_2 + i_- \begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - r(B_1) - r(B_2), \quad (5.43)$$

$$\max_{X_3 \in \mathcal{V}_3} i_+(X_3) = n_2 + i_+ \begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - r[A, B], \quad (5.44)$$

$$\min_{X_3 \in \mathcal{V}_3} i_+(X_3) = 0, \quad (5.45)$$

$$\min_{X_3 \in \mathcal{V}_3} i_-(X_3) = i_- \begin{bmatrix} A & B_1 \\ B_1^* & 0 \end{bmatrix} - r(B_1). \quad (5.46)$$

A further work is to give the extremal ranks and inertias of the $A_1 - B_1 X B_1^*$ subject to the common Hermitian solution of the $k - 1$ consistent LMIs

$$B_2 X B_2^* \succcurlyeq A_2, \dots, B_k X B_k^* \succcurlyeq A_k,$$

and to establish necessary and sufficient condition for the set of LMIs

$$B_1 X B_1^* \succcurlyeq A_1, B_2 X B_2^* \succcurlyeq A_2, \dots, B_k X B_k^* \succcurlyeq A_k,$$

to have a common Hermitian solution.

Finally, it should be pointed out that the rank and inertia of a matrix, as two simplest concepts in linear algebra, are also one of the richest fields in mathematics that admit ten thousands of analytical formulas.

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